February 13, 2018 (UIUC-AG seminar)

Noncommutative McKay correspondence $\boxed{\text{pretalk}}$

A lot of what I do generalizes results in commutative algebra & in Algebraic Geometry to the noncommutative/quantum setting.

Classic Setting

affine space $\mathbb{A}^n_{\mathbb{C}}$

$\updownarrow$

Coordinate ring of $\mathbb{A}^n_{\mathbb{C}}$
$= \mathbb{C}[v_1, \ldots, v_n]$

symmetries of $\mathbb{A}^n_{\mathbb{C}}$
= automorphisms of $\mathbb{C}[\underline{v}]$ from group actions

parts of $\mathbb{A}^n_{\mathbb{C}}$ remaining fixed under symmetries $\{\mathbb{A}^n /\!/ G\}$
= invariant ring $\mathbb{C}[\underline{v}]^G$ for G-action on $\mathbb{C}[\underline{v}]$

Noncom/Quantum Setting

"quantum affine space"

$\updownarrow$

noncom. graded algebra A that behaves ring theoretically & homologically like $\mathbb{C}[v_1, \ldots, v_n]$
= coordinate ring of q. aff. space

group, and more generally actions of Hopf algebras on A
= "quantum symmetries"

"quantum affine space $/\!/$ Hopf algs"

{largely unexplored}

= A^H invariant ring

{well-studied}

...

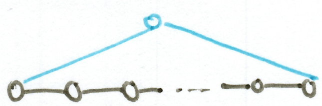
In today's talk, the specific aim is to generalize the classic McKay corresp. to the noncommutative setting

Classic Setting

Several algebraic, geometric, rep-theoretic notions that correspond to

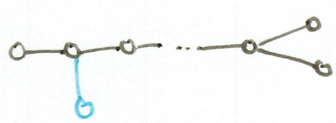
$\boxed{\text{ADE Dynkin diagrams}}$
extended

type A: $\tilde{A}_n$



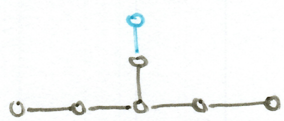
← n vertices

type D: $\tilde{D}_n$

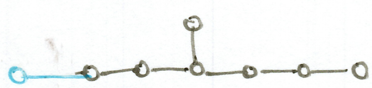


← n vertices

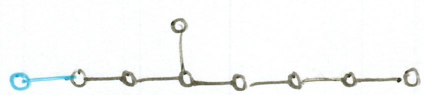
type E: $\tilde{E}_6$



$\tilde{E}_7$



$\tilde{E}_8$



For now, let's focus on one part of the corresp.:
Start with $G \leq SL_2(\mathbb{C})$ that acts linearly & faithfully on $\mathbb{C}[u,v]$

$\boxed{\text{(Simple modules / irreducible reps of) FINITE GROUP } G}$ $\overset{1\text{-}1}{\longleftrightarrow}$ $\boxed{\text{ADE diagram}}$

$\longmapsto$ "McKay quiver"

We'll see that in

Noncom setting

One gets

$\boxed{\text{ADE diagrams}}$

+

$\boxed{\text{more diagrams}}$

Let's consider

$$G = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \leq SL_2(\mathbb{C})$$
$$\cong \mathbb{Z}_2$$

acts on $A = \dfrac{\mathbb{C}\langle u,v\rangle}{(uv+vu)}$

by swapping gens u & v

this is a noncom alg. that behaves a lot like $\mathbb{C}[u,v]$

There are five families of subgroups G of $SL_2(\mathbb{C})$ up to isomorphism; they act on $\mathbb{C}[u,v]$ and were studied extensively by F. Klein, 1884.

J. McKay (1980) used reps G to construct a directed graph (quiver) for which its isomorphism class corresponds :

McKay quiver:

vertices corresponding to set of non-isomorphic simple modules of G

[# of vertices $= |G| - 1$]

[# of vertices of extended ver $= |G|$]

$\{V_0, V_1, \ldots, V_d\}$

$\uparrow$ for trivial module

arrows $V_i \xrightarrow{m_{ij}} V_j$

[replace 2-cycles with edge]

if $V_j \otimes \Big[\underbrace{\text{fundamental rep } \mathbb{C}u \oplus \mathbb{C}v}_{\text{Fund}}\Big] = V_i^{\oplus m_{ij}} \oplus \sim$

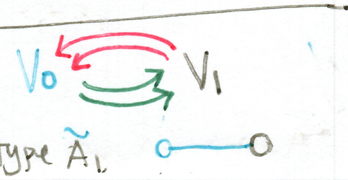
Ex. $G = \langle g \mid g^2 = 1 \rangle = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \le SL_2(\mathbb{C})$ acts on $\mathbb{C}[u,v]$

$V_0 =$ triv rep $= \mathbb{C}x_0$, $g \cdot x_0 = x_0$. $V_1 =$ sign rep $= \mathbb{C}x_1$, $g \cdot x_1 = -x_1$

Fund rep: $\mathbb{C}u \oplus \mathbb{C}v$, $g \cdot u = -u$, $g \cdot v = -v$.

$\boxed{V_0 \otimes \text{Fund} = V_1 \oplus V_1}$

$\boxed{V_1 \otimes \text{Fund} = V_0 \oplus V_0}$



$G = \langle g \mid g^2 = 1 \rangle = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \le SL_2(\mathbb{C})$

acts on

$$\frac{\mathbb{C}\langle u,v \rangle}{(uv + vu)}$$

linearly & faithfully

$V_0 = \mathbb{C}x_0$, $g \cdot x_0 = x_0$ (triv)

$V_1 = \mathbb{C}x_1$, $g \cdot x_1 = -x_1$ (sign)

Fund $= \mathbb{C}u \oplus \mathbb{C}v$

$g \cdot u = v$, $g \cdot v = u$

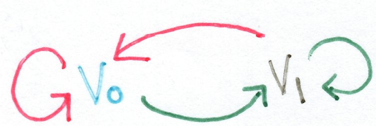
also by change of basis:

$g \cdot (u+v) = u+v$

$g \cdot (u-v) = -(u-v)$

So, $\boxed{V_0 \otimes \text{Fund} = V_0 \oplus V_1}$

$\boxed{V_1 \otimes \text{Fund} = V_0 \oplus V_1}$



yields

NOT ADE diagram, type $\tilde{L}_1$ (New!)

more in seminar this afternoon...